TRANSVERSE MERCATOR AND LAMBERT CONFORMAL CONIC MAP PROJECTION FUNCTIONS

Alan Vonderohe (February 2020)

Contents

1. Background.

A map projection is a mathematical surface with functional relationships between points having geodetic coordinates (latitude and longitude) on a reference ellipsoid and corresponding points with two-dimensional rectangular coordinates (northing and easting) on the map projection surface. A map projection is described by mathematical transformations between the two types of coordinates. A “direct” transformation computes northing and easting (N,E) from latitude and longitude (ϕ,λ):

$$N = f_1(\phi,\lambda, \text{ellipsoid parameters, map projection parameters})$$
$$E = f_2(\phi,\lambda, \text{ellipsoid parameters, map projection parameters})$$

An “inverse” transformation computes ϕ,λ from N,E:

$$\phi = g_1(N,E, \text{ellipsoid parameters, map projection parameters})$$
$$\lambda = g_2(N,E, \text{ellipsoid parameters, map projection parameters})$$

In the equations above, ellipsoid parameters are two descriptors that define the size and shape of the reference ellipsoid. In this document, those two parameters are the semi-major axis (a) and the semi-minor axis (b). Map projection parameters are descriptors that define the size and shape of the map projection surface and its location and orientation with respect to the reference ellipsoid. Map projection parameters also define the location of the rectangular coordinate origin and its false northing and false easting values.

On any map projection, each point has a scale factor and a convergence. On conformal map projections, such as transverse Mercator and Lambert conformal conic, the scale factor is the same in all directions at any given point but is variable from point to point. Convergence, sometimes referred to as “the mapping angle”, also varies from point to point and is the angle between geodetic north and grid north. It is defined as a geodetic azimuth minus the projection of that azimuth on the map projection coordinate grid. Convergence at a point is not the difference between geodetic azimuth and grid azimuth. Such a difference depends not only upon convergence but also upon the “arc-to-chord” or “second term” correction. Arc-to-chord corrections, at any point, vary with distance and direction to an arbitrary second point, whose coordinates must be specified. This document presents methods for computing scale and convergence. It does not address methods for computing arc-to-chord corrections. For many applications arc-to-chord corrections are negligible.

The scale factor (denoted k) and convergence (denoted γ) are found by functions:

$$k = h_1(\phi, \lambda, a, b, \text{map projection parameters})$$
$$\gamma = h_2(\phi, \lambda, a, b, \text{map projection parameters})$$

Finally, any given point on a map projection has a linear distortion that is the ratio of a very small distance on Earth's surface to the corresponding very small distance on the map projection surface. Linear distortion accounts for the separation between the two surfaces and expresses the error to be encountered if ground distances are used for grid distances:

$$LD = i(\phi, a, b, k, \text{ellipsoid height})$$

This document presents each of the functions described above for transverse Mercator projections and two types of Lambert conformal conic projections.

2. General Notation and Definitions.

- ϕ Geodetic latitude, positive north.
- λ Geodetic longitude, positive east (0° to 360°).
- N Northing coordinate on the projection.
- E Easting coordinate on the projection.
- λ_o Longitude of the central meridian and the coordinate origin.
- E_o False easting of the coordinate origin.
- ϕ_o Latitude of the coordinate origin (transverse Mercator and non-intersecting Lambert conformal conic). Also, latitude of the central parallel (Lambert conformal conic).
- k_o Scale factor along the central meridian (transverse Mercator) or central parallel (Lambert conformal conic).
- k Scale factor.
- γ Convergence.
- a Semi-major axis of the reference ellipsoid.
- b Semi-minor axis of the reference ellipsoid.
- e First eccentricity of the reference ellipsoid.
- e' Second eccentricity of the reference ellipsoid.
- v Radius of curvature in the prime vertical.

ρ Radius of curvature in the meridian.
R_G Gaussian or geometric mean radius of curvature.
N_g Geoid height.
h Ellipsoid height.
H Orthometric height.

3. Ellipsoid Constants.

$$e^2 = \frac{a^2 - b^2}{a^2}$$

$$e'^2 = \frac{e^2}{1 - e^2}$$

$$n = \frac{a - b}{a + b}$$

4. Transverse Mercator Projections (After Stem (1989)).

Transverse Mercator projections are based upon right cylinders whose axes lie in the equatorial plane and pass through the center of the reference ellipsoid. The selected right cylinder can be secant or tangent to the reference ellipsoid. It can also not intersect the reference ellipsoid at all.

4.a. Projection-Specific Notation.

<u>NOTE: Projection parameters are ϕ_o, λ_o, k_o, N_o, E_o.</u>

ω Rectifying latitude.
S Meridional distance from the equator, multiplied by k_o.
ω_o Rectifying latitude at ϕ_o.
S_o Meridional distance from the equator to ϕ_o, multiplied by k_o.
r Radius of the rectifying sphere.
N_o False northing of the coordinate origin.

4.b. Projection Constants.

$$r = a(1-n)(1-n^2)\left(1+\frac{9n^2}{4}+\frac{225n^4}{64}\right)$$

$$u_2 = -\frac{3n}{2}+\frac{9n^3}{16}$$

$$u_4 = \frac{15n^2}{16}-\frac{15n^4}{32}$$

$$u_6 = -\frac{35n^3}{48}$$

$$u_8 = \frac{315n^4}{512}$$

$$U_0 = 2(u_2 - 2u_4 + 3u_6 - 4u_8)$$
$$U_2 = 8(u_4 - 4u_6 + 10u_8)$$
$$U_4 = 32(u_6 - 6u_8)$$
$$U_6 = 128u_8$$

$$v_2 = \frac{3n}{2}-\frac{27n^3}{32}$$

$$v_4 = \frac{21n^2}{16}-\frac{55n^4}{32}$$

$$v_6 = \frac{151n^3}{96}$$

$$v_8 = \frac{1097n^4}{512}$$

$$V_0 = 2(v_2 - 2v_4 + 3v_6 - 4v_8)$$
$$V_2 = 8(v_4 - 4v_6 + 10v_8)$$
$$V_4 = 32(v_6 - 6v_8)$$
$$V_6 = 128v_8$$

$$\omega_o = \phi_o + \sin\phi_o\cos\phi_o(U_0 + U_2\cos^2\phi_o + U_4\cos^4\phi_o + U_6\cos^6\phi_o)$$
$$S_o = k_o\omega_o r$$

4.c. Direct Transformation (ϕ,λ to N,E)

$$L = -(\lambda - \lambda_o)cos\phi$$
$$\omega = \phi + sin\phi cos\phi(U_o + U_2cos^2\phi + U_4cos^4\phi + U_6cos^6\phi)$$
$$S = k_o\omega r$$

$$t = \tan\phi$$

$$\eta^2 = e'^2 \cos^2\phi$$

$$R = \frac{k_o a}{\sqrt{1 - e^2 \sin^2\phi}}$$

$$A_1 = -R$$

$$A_2 = \frac{Rt}{2}$$

$$A_3 = \frac{1 - t^2 + \eta^2}{6}$$

$$A_4 = \frac{5 - t^2 + \eta^2(9 + 4\eta^2)}{12}$$

$$A_5 = \frac{5 - 18t^2 + t^4 + \eta^2(14 - 58t^2)}{120}$$

$$A_6 = \frac{61 - 58t^2 + t^4 + \eta^2(270 - 330t^2)}{360}$$

$$A_7 = \frac{61 - 479t^2 + 179t^4 - t^6}{5040}$$

$$N = S - S_o + N_o + A_2 L^2 \left[1 + L^2 (A_4 + A_6 L^2)\right]$$

$$E = E_o + A_1 L \left\{1 + L^2 \left[A_3 + L^2 (A_5 + A_7 L^2)\right]\right\}$$

4.d. Inverse Transformation (N,E to φ,λ)

$$\omega = \frac{N - N_o + S_o}{k_o r}$$

$$\phi_f = \omega + \sin\omega \cos\omega (V_0 + V_2 \cos^2\omega + V_4 \cos^4\omega + V_6 \cos^6\omega)$$

$$t_f = \tan\phi_f$$

$$\eta_f^2 = e'^2 \cos^2\phi_f$$

$$R_f = \frac{k_o a}{\sqrt{1 - e^2 \sin^2\phi_f}}$$

$$Q = \frac{E - E_o}{R_f}$$

$$B_2 = \frac{-t_f (1 + \eta_f^2)}{2}$$

$$B_3 = \frac{-(1+2t_f^2+\eta_f^2)}{6}$$

$$B_4 = \frac{-(5+3t_f^2+\eta_f^2(1-9t_f^2)-4\eta_f^4)}{12}$$

$$B_5 = \frac{5+28t_f^2+24t_f^4+\eta_f^2(6+8t_f^2)}{120}$$

$$B_6 = \frac{61+90t_f^2+45t_f^4+\eta_f^2(46-252t_f^2-90t_f^4)}{360}$$

$$B_7 = \frac{-(61+662t_f^2+1320t_f^4+720t_f^6)}{5040}$$

$$L = -Q\{1+Q^2[B_3+Q^2(B_5+B_7Q^2)]\}$$

$$\phi = \phi_f + B_2Q^2[1+Q^2(B_4+B_6Q^2)]$$

$$\lambda = \lambda_o - L/cos\phi_f$$

4.e. Scale and Convergence.

<u>Convergence (γ) in radians.</u>

$$t = \tan\phi$$

$$\eta^2 = e'^2\cos^2\phi$$

$$L = -(\lambda-\lambda_o)cos\phi$$

$$C_1 = -t$$

$$C_3 = \frac{1}{3}(1+3\eta^2+2\eta^4)$$

$$C_5 = \frac{1}{15}(2-t^2)$$

$$\gamma = C_1L\left[1+L^2(C_3+C_5L^2)\right]$$

<u>Scale Factor (k).</u>

$$F_2 = \frac{1}{2}(1+\eta^2)$$

$$F_4 = \frac{1}{12}\left[5-4t^2+\eta^2(9-24t^2)\right]$$

$$k = k_o\left[1+F_2L^2(1+F_4L^2)\right]$$

5. Lambert Conformal Conic Projections.

Lambert conformal conic projections are based upon right circular cones whose axes coincide with the minor axis the reference ellipsoid. The selected right circular cone can be secant or tangent to the reference ellipsoid. It can also not intersect the reference ellipsoid at all.

5.a. Secant (Two Standard Parallels) (After Stem (1989)).

5.a.i. Projection-Specific Notation.

<u>NOTE: Projection parameters are ϕ_N, ϕ_S, ϕ_b, λ_o, N_b, E_o.</u>

ϕ_N	Latitude of the northern standard parallel.
ϕ_S	Latitude of the southern standard parallel.
ϕ_b	Latitude of the coordinate origin.
N_b	False northing of the coordinate origin.
R	Mapping radius at latitude ϕ.
R_b	Mapping radius at latitude ϕ_b.

5.a.ii. Projection Constants.

$$Q_S = \frac{1}{2}\left[\ln\left(\frac{1+\sin\phi_S}{1-\sin\phi_S}\right) - e\ln\left(\frac{1+e\sin\phi_S}{1-e\sin\phi_S}\right)\right]$$

$$W_S = \sqrt{1-e^2\sin^2\phi_S}$$

$$Q_N = \frac{1}{2}\left[\ln\left(\frac{1+\sin\phi_N}{1-\sin\phi_N}\right) - e\ln\left(\frac{1+e\sin\phi_N}{1-e\sin\phi_N}\right)\right]$$

$$W_N = \sqrt{1-e^2\sin^2\phi_N}$$

$$Q_b = \frac{1}{2}\left[\ln\left(\frac{1+\sin\phi_b}{1-\sin\phi_b}\right) - e\ln\left(\frac{1+e\sin\phi_b}{1-e\sin\phi_b}\right)\right]$$

$$\phi_o = \sin^{-1}\left[\frac{\ln\left(\frac{W_N\cos\phi_S}{W_S\cos\phi_N}\right)}{Q_N - Q_S}\right]$$

$$K = \frac{a\cos\phi_S \exp\left(Q_S \sin\phi_o\right)}{W_S \sin\phi_o}$$

$$R_b = \frac{K}{\exp\left(Q_b \sin\phi_o\right)}$$

5.a.iii. Direct Transformation (Φ,λ to N,E).

$$Q = \frac{1}{2}\left[ln\left(\frac{1+sin\phi}{1-sin\phi}\right) - eln\left(\frac{1+esin\phi}{1-esin\phi}\right)\right]$$

$$R = \frac{K}{\exp\left(Q sin\phi_o\right)}$$

$$\gamma = -(\lambda_o - \lambda) sin\phi_o$$

$$N = R_b + N_b - R\cos\gamma$$
$$E = E_o + R\sin\gamma$$

$$k = \frac{(Rsin\phi_o)\sqrt{1-e^2sin^2\phi}}{acos\phi}$$

5.a.iv. Inverse Transformation (N,E to Φ,λ).

$$R' = R_b - N + N_b$$

$$E' = E - E_o$$

$$\gamma = tan^{-1}\left(\frac{E'}{R'}\right)$$

$$\lambda = \lambda_o + \frac{\gamma}{sin\phi_o}$$

$$R = \sqrt{R'^2 + E'^2}$$

$$Q = \frac{ln\left(\frac{K}{R}\right)}{sin\phi_o}$$

Latitude computation is iterative:

1. Approximate ϕ with $\phi = \sin^{-1}\left(\frac{\exp(2Q)-1}{\exp(2Q)+1}\right)$
2. Compute a correction of ($-f_1$ / f_2) from:

   $$f_1 = \frac{1}{2}\left[\ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right) - e\ln\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)\right] - Q$$

   $$f_2 = \frac{1}{1-\sin^2\phi} - \frac{e^2}{1-e^2\sin^2\phi}$$

3. Add the correction to sinϕ.
4. Repeat steps 2 and 3 twice more for a total of three corrections.

$$k = \frac{(Rsin\phi_o)\sqrt{1 - e^2 sin^2\phi}}{acos\phi}$$

5.b. Non-Intersecting (Central Parallel and Its Scale Factor) (After Bomford (1985)).

5.b.i. Projection-Specific Notation.

NOTE: Projection parameters are ϕ_o, λ_o, k_o, N_o, E_o.

N_o False northing of the coordinate origin and northing of the central parallel at the central meridian.
r_o Mapping radius at the coordinate origin.
v_o Radius of curvature in the prime vertical at ϕ_o.
ρ_o Radius of curvature in the meridian at ϕ_o.
M_o Meridional arc from the equator to ϕ_o.
M_Φ Meridional arc from the equator to ϕ.
M Meridional arc from ϕ to ϕ_o (M is negative if $M_\Phi < M_o$).
s Meridional arc from ϕ to ϕ_o scaled to the projection (s is negative if $M_\Phi < M_o$).

5.b.ii. Projection Constants.

$$v_o = \frac{a}{\sqrt{1 - e^2 sin^2 \phi_o}}$$

$$\rho_o = \frac{a(1 - e^2)}{(1 - e^2 sin^2 \phi_o)^{\frac{3}{2}}}$$

$$r_o = \frac{k_o v_o}{tan \phi_o}$$

$$A_o = 1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256}$$

$$A_2 = \frac{3}{8}\left(e^2 + \frac{e^4}{4} + \frac{15e^6}{128}\right)$$

$$A_4 = \frac{15}{256}\left(e^4 + \frac{3e^6}{4}\right)$$

$$A_6 = \left(\frac{35}{3072}\right)e^6$$

$$M_o = a(A_0\phi_o - A_2 \sin 2\phi_o + A_4 \sin 4\phi_o - A_6 \sin 6\phi_o)$$

5.b.iii. Direct Transformation (Φ,λ to N,E).

$$M_\phi = a(A_0\phi - A_2 \sin 2\phi + A_4 \sin 4\phi - A_6 \sin 6\phi)$$

$$M = M_\phi - M_o$$

$$M_3 = \frac{M^3}{6\rho_o \nu_o}$$

$$M_4 = \frac{M^4 \tan\phi_o (1 - 4e'^2 \cos^2 \phi_o)}{24\rho_o \nu_o^2}$$

$$M_5 = \frac{M^5 (5 + 3\tan^2 \phi_o - 3e'^2 - e'^2 \cos^2 \phi_o)}{120\rho_o^2 \nu_o^2}$$

$$M_6 = \frac{M^6 \tan\phi_o (7 + 4\tan^2 \phi_o)}{240\rho_o^2 \nu_o^3}$$

$$M_7 = \frac{M^7 (60\tan^4 \phi_o + 180\tan^2 \phi_o + 61)}{5040\rho_o^3 \nu_o^3}$$

$$s = k_o(M + M_3 + M_4 + M_5 + M_6 + M_7)$$

$$\gamma = (\lambda - \lambda_o)\sin\phi_o$$

$$\Delta E = (r_o - s)\sin\gamma$$

$$E = E_o + \Delta E$$

$$N = N_o + s + \Delta E \tan\left(\frac{\gamma}{2}\right)$$

5.b.iv. Inverse Transformation (N,E to Φ,λ).

$$\gamma = \tan^{-1}\left(\frac{E - E_o}{r_o - N + N_o}\right)$$

$$\lambda = \left(\frac{\gamma}{\sin\phi_o}\right) + \lambda_o$$

$$s = N - N_o - (E - E_o)\tan\left(\frac{\gamma}{2}\right)$$

Computation of M is iterative:

1. Begin with an approximation of $M_a = \frac{s}{k_o}$.
2. Compute the following:

$$M_{3_a} = \frac{M_a^3}{6\rho_o \nu_o}$$

$$M_{4_a} = \frac{M_a^4 \tan\phi_o (1 - 4e'^2 \cos^2\phi_o)}{24\rho_o \nu_o^2}$$

$$M_{5_a} = \frac{M_a^5 (5 + 3\tan^2\phi_o - 3e'^2 - e'^2 \cos^2\phi_o)}{120\rho_o^2 \nu_o^2}$$

$$M_{6_a} = \frac{M_a^6 \tan\phi_o (7 + 4\tan^2\phi_o)}{240\rho_o^2 \nu_o^3}$$

$$M_{7_a} = \frac{M_a^7 (60\tan^4\phi_o + 180\tan^2\phi_o + 61)}{5040\rho_o^3 \nu_o^3}$$

$$-F_a = \left(\frac{s}{k_o}\right) - M_a - M_{3_a} - M_{4_a} - M_{5_a} - M_{6_a} - M_{7_a}$$

$$\left(\frac{\partial F}{\partial M}\right)_a = 1 + \frac{3M_{3_a}}{M_a} + \frac{4M_{4_a}}{M_a} + \frac{5M_{5_a}}{M_a} + \frac{6M_{6_a}}{M_a} + \frac{7M_{7_a}}{M_a}$$

$$\delta M = \frac{-F_a}{\left(\frac{\partial F}{\partial M}\right)_a}$$

3. Add δM to M_a.
4. Repeat steps 2 and 3 until the absolute value of δM is less than 0.00005 (meters).

$$M = M_a$$

$$M_\phi = M_o + M$$

Computation of latitude is iterative:

1. Begin with an approximation of $\phi_a = \frac{M_\phi}{aA_o}$ (radians).
2. Compute the following:

$$-G_a = M_\phi - a(A_0\phi_a - A_2 \sin 2\phi_a + A_4 \sin 4\phi_a - A_6 \sin 6\phi_a)$$

$$\left(\frac{\partial G}{\partial \phi}\right)_a = a(A_0 - 2A_2 \cos 2\phi_a + 4A_4 \cos 4\phi_a - 6A_6 \cos 6\phi_a)$$

$$\delta\phi = \frac{-G_a}{\left(\dfrac{\partial G}{\partial \phi}\right)_a}$$

3. Add $\delta\phi$ to ϕ_a.
4. Repeat steps 2 and 3 until the absolute value of $\delta\phi$ is less than 0.0000005 seconds of arc.

$$\phi = \phi_a$$

5.b.v. Scale and Convergence.

Convergence (γ) in Angular Units.

$$\gamma = (\lambda - \lambda_o)\sin\phi_o$$

Scale Factor (k).

$$v_o = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi_o}}$$

$$\rho_o = \frac{a(1-e^2)}{(1-e^2\sin^2\phi_o)^{\frac{3}{2}}}$$

$$M_\phi = a(A_0\phi - A_2 \sin 2\phi + A_4 \sin 4\phi - A_6 \sin 6\phi)$$

$$M = M_\phi - M_o$$

$$J_1 = \frac{M^2}{2\rho_o \upsilon_o}$$

$$J_2 = \frac{M^3 \tan\phi_o}{6\rho_o \upsilon_o^2}$$

$$J_3 = \frac{M^4(5 + 3\tan^2\phi_o)}{24\rho_o^2 \upsilon_o^2}$$

$$k = k_o(1 + J_1 + J_2 + J_3)$$

6. Linear Distortion at ϕ,λ.

$$\nu = \frac{a}{\sqrt{1 - e^2 sin^2 \phi}}$$

$$\rho = \frac{a(1-e^2)}{(1 - e^2 sin^2 \phi)^{\frac{3}{2}}}$$

$$R_G = \sqrt{\rho v} = \frac{a\sqrt{1-e^2}}{1-e^2 sin^2\phi}$$
$$h = H + N_g$$
$$LD = k\left(\frac{R_G}{R_G + h}\right) - 1$$

For linear distortion in nearest integer parts per million:

$$LD_{ppm} = int\left[round\left(LD * 10^6, 0\right)\right]$$

7. Geocentric Coordinates.

Geocentric coordinates are 3D right-handed Cartesian with the origin at the intersection of the equatorial plane and the minor axis of the reference ellipsoid. The positive X axis lies in the equatorial plane and passes through the Greenwich meridian (0° longitude). The positive Y axis lies in the equatorial plane and passes through the meridian at 90°east longitude. The positive Z axis passes through the pole at 90° north latitude. Relationships among geocentric coordinates and geodetic coordinates are shown in Figure 1, where p is a point outside the reference ellipsoid and h_p is the ellipsoid height of p (positive from the ellipsoid surface towards p).

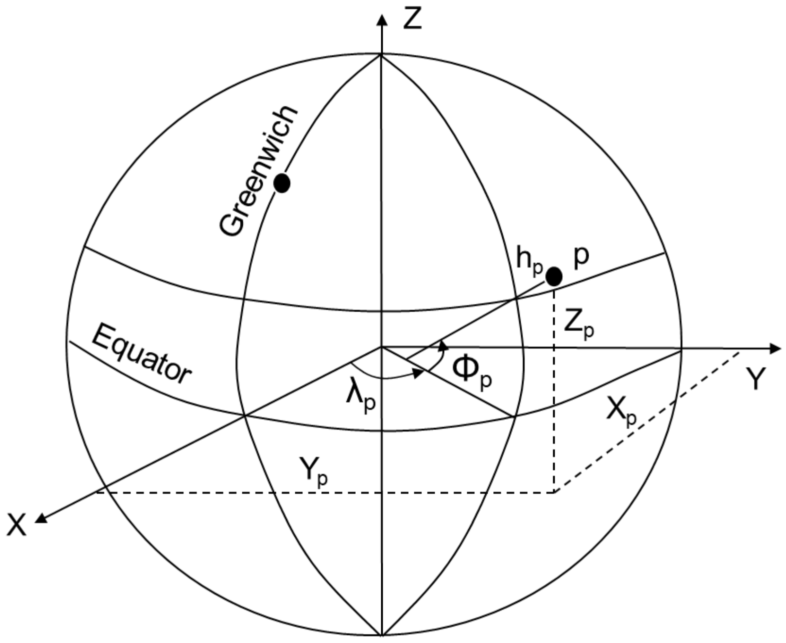

Figure 1.
Geocentric and Geodetic Coordinates of Point P

GNSS satellites are tracked in geocentric coordinates and transformations between geocentric and geodetic coordinates are essential.

7.a. Direct Transformation (Φ,λ,h to X,Y,Z).

$$\nu = \frac{a}{\sqrt{1 - e^2 sin^2\phi}}$$

$$X = (\nu + h)cos\phi cos\lambda$$

$$Y = (\nu + h)cos\phi sin\lambda$$

$$Z = [\nu(1 - e^2) + h]sin\phi$$

7.b. Inverse Transformation (X,Y,Z to Φ,λ,h).

$$\lambda = sin^{-1}\left(\frac{Y}{\sqrt{X^2 + Y^2}}\right) = cos^{-1}\left(\frac{X}{\sqrt{X^2 + Y^2}}\right)$$

Computation of latitude is iterative:

1. Begin with an approximation of

$$\phi_a = tan^{-1}\left(\frac{Z}{\sqrt{X^2 + Y^2}}\right)$$

2. Compute the following:

$$\nu_a = \frac{a}{\sqrt{1 - e^2 sin^2\phi_a}}$$

$$\phi = tan^{-1}\left(\frac{Z + \nu_a e^2 sin\phi_a}{\sqrt{X^2 + Y^2}}\right)$$

$$\phi_a = \phi$$

Repeat step 2 until the difference between ϕ and ϕ_a is less than 0.0000005 seconds of arc.

$$h = \frac{\sqrt{X^2 + Y^2}}{cos\phi} - \nu$$

8. List of References.

Bomford, G., (1985), Geodesy, 4^{th} Edition, Oxford University Press, New York, 855 pp.

Stem, J., (1989), State Plane Coordinate Systems of 1983, NOAA Manual NOS NGS 5, Rockville, MD.