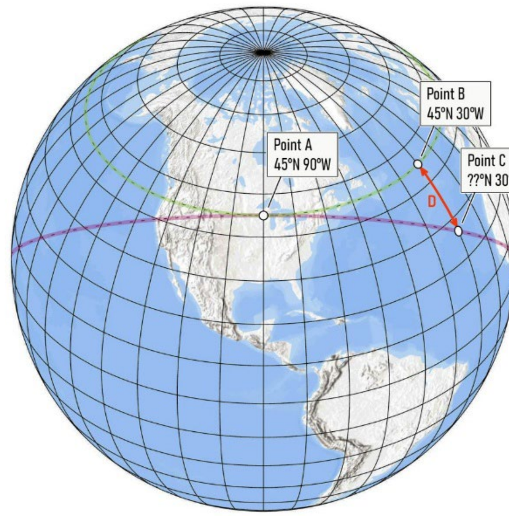
Answer to Surveying Challenge Problem

William Rose, 20260218

Challenge posted by Howard Veregin, May 2, 2024: "Which Way is East?", May 2, 2024, https://www.sco.wisc.edu/2024/05/02/which-way-is-east-a-reader-challenge/, retrieved 2026-02-18

"Your challenge, if you choose to accept it, is to determine the equations that can be used to compute the deviation (D) along a meridian between a parallel and a great circle arc tangent to that parallel at a point. An example is shown by the red arrow labeled D in Figure 1. However, your method must be general enough to use for a) any parallel and b) any meridian."



Answer

The deviation or offset may be expressed as an angle, or as a linear distance. If angle is desired:

$$\Delta\phi_{exact} = \phi - \tan^{-1}(\tan\phi \cos\Delta\lambda)$$
(1a)

$$\Delta\phi_{approx} \cong \frac{1}{2}\frac{\tan\phi}{(\tan^2\phi + 1)}\Delta\lambda^2$$
(1b)

where ϕ=geodetic latitude of the initial point and $\Delta\lambda$=angular distance along the great circle that bears east at the initial point. Equations 1a and 1b are obtained using an equation from spherical trigonometry and the Taylor series for cosine and inverse tangent, explained in detail below. See equations 6 and 9 below.

The deviation, or offset, is converted from angle to linear distance by multiplying by the radius:

$$D = R_\phi \Delta\phi$$
(2)

where R_ϕ=Earth radius at latitude ϕ, and $\Delta\lambda$ is obtained from equation 1a or 1b.

Comparison To 1973 *Manual Of Surveying Instructions*

The challenge problem includes Fig. 13 from the 1973 *Manual of Surveying Instructions*. Figure 13 includes the equation

$$\text{Offset (in chains)} = \frac{1}{R_p} \cdot \frac{(m\phi)^2}{2} \cdot \sin b \tag{3}$$

where mϕ=linear distance along the great circle track, b=initial bearing of the great circle track, and R_p is a constant that depends on latitude, given in the table on p.56 on the 1973 Manual. Since b=90° in the challenge problem, sin b=1. Comparison of equation 3 above with equation 1b and 2 in the approximate solution above indicates that

$$\frac{1}{R_p}\ (\text{1973 Manual}) = kR_\phi \frac{\tan\phi}{(\tan^2\phi + 1)}$$
$$\frac{1}{R_p} = \frac{R_\phi}{r_\phi^2} \frac{\tan\phi}{(\tan^2\phi + 1)} \tag{4}$$

where $k = 1/r_\phi^2$ is an adjustment to convert squared distance $(m\phi)^2$ to squared angle $\Delta\lambda^2$, and R_ϕ is the geocentric radius at latitude ϕ:

$$R_\phi = R_{eq} \sqrt{\frac{(1-e^2)^2 \sin^2\phi + \cos^2\phi}{1 - e^2 \sin^2\phi}} \tag{5}$$

and $r_\phi = R_\phi \cos\phi$ = distance from the surface to the Earth's axis.

Table 1 shows values for $1/R_p$ at different latitudes, from the 1973 *Manual* and computed with equation 4. The computed values of $1/R_p$ use R_ϕ and r_ϕ expressed in chains, to match the 1973 *Manual*. R_{eq}=3963.19 miles, ϕ=45.575°, e^2=6.69438e-3. See surveyChallenge.xlsx.

The computed values of $1/R_p$ match the values from the *Manual*.

Table 1.

Latitude	$1/R_p$ from 1973 *Manual*	$1/R_p$ from eq.4
30	1.819E-06	1.819E-06
40	2.643E-06	2.643E-06
50	3.751E-06	3.751E-06
60	5.449E-06	5.449E-06
70	8.640E-06	8.640E-06

Units for $1/R_p$ = 1/chains.

Table 2 shows the values for offset D, in links, from the 1973 *Manual*, and computed with equations 1 and 2, in surveyChallenge.xlsx. Note the excellent agreement between D_{Manual}, D_{exact}, and D_{approx}.

Table 2.

L=distance along great circle	D_{Manual} from 1973 *Manual*	Angle $\Delta\lambda$	D_{exact} from eqs.1a,2	D_{approx} from eqs.1b,2
0	0	0.000E+00	0.0	0.0
80	1	3.605E-04	1.0	1.0
160	4	7.209E-04	4.1	4.1
240	9	1.081E-03	9.3	9.3
320	16.5	1.442E-03	16.4	16.4
400	25.5	1.802E-03	25.7	25.7
480	37	2.163E-03	37.0	37.0

Units: L in chains; D_{Manual}, D_{exact}, D_{approx} in links; $\Delta\lambda$ in radians. The values for D from the Manual appear to have been rounded to the nearest half-link.

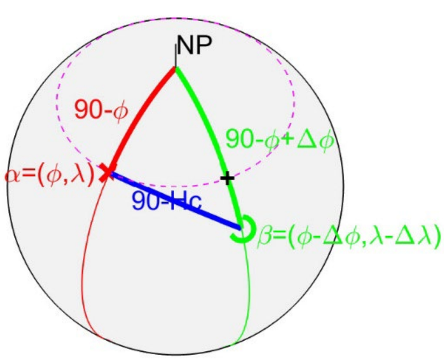
Solution Details

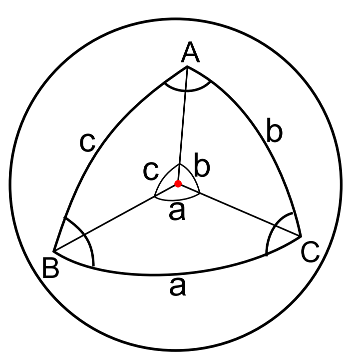
This is a spherical triangle problem, as seen in the figure at left below. Each side of the triangle is a great circle arc. The corners of the triangle are the start point α, the end point β, and the north pole. The (latitude,longitude) coordinates of the start and end points are

$$\alpha = (\phi, \lambda)$$

$$\beta = (\phi - \Delta\phi, \lambda - \Delta\lambda)$$

The linear distance D in the figure from the online challenge is the distance from the end point β to the black + symbol in the figure below. This linear distance corresponds to arc length $\Delta\phi$ in the figure at left below. We want a formula for $\Delta\phi$ as a function of $\Delta\lambda$. We associate our spherical triangle with a general spherical triangle ABC shown on the right below.





We know vertex angles A=$\Delta\lambda$ and B=90°. We know side c=90-ϕ. We wish to find side b=90-ϕ+$\Delta\phi$. We use the identity

$$\cos c\, \cos A = \cot b\, \sin c - \cot B\, \sin A$$

Substitute A=$\Delta\lambda$, B=90°, c=90-ϕ, b=90-ϕ+$\Delta\phi$. Remember cot(B)=cot(90°)=0.

$$\cos(90-\phi)\cos(\Delta\lambda) = \cot(90-\phi+\Delta\phi)\sin(90-\phi)$$

Remember cos(90-ϕ)=sin(ϕ), sin(90-ϕ)=cos(ϕ), cot(90-ϕ+$\Delta\phi$)=tan(ϕ-$\Delta\phi$):

$$\sin\phi\cos\Delta\lambda = \tan(\phi-\Delta\phi)\cos\ \phi$$

Therefore

$$\tan(\phi-\Delta\phi) = \frac{\sin\phi\cos\Delta\lambda}{\cos\ \phi}$$

$$\Delta\phi = \phi - \tan^{-1}(\tan\phi\cos\Delta\lambda) \quad \text{(angle)} \tag{6}$$

Equation 6 is a general solution to the challenge, returning the offset as an angle of arc.

We get an approximate solution, which is very accurate good when the distance run along the arc is less than 1°, by applying the Taylor series expansion for cos $\Delta\lambda$: cos $\Delta\lambda \approx 1-\Delta\lambda^2/2$. Therefore we have

$$\Delta\phi \cong \phi - \tan^{-1}\left(\tan\phi\left(1-\frac{\Delta\lambda^2}{2}\right)\right)$$

$$\Delta\phi \cong \phi - \tan^{-1}\left(\tan\phi - \frac{\Delta\lambda^2}{2}\tan\phi\right) \tag{7}$$

Apply the Taylor series expansion for $\tan^{-1}(\theta+\Delta\theta)$:

$$\tan^{-1}(\theta+\Delta\theta) \cong \tan^{-1}\theta + \frac{1}{\theta^2+1}\Delta\theta - \frac{\theta}{(\theta^2+1)^2}\Delta\theta^2$$

Substitute $\theta = \tan\phi$, $\Delta\theta = -\frac{\Delta\lambda^2}{2}\tan\phi$:

$$\tan^{-1}\left(\tan\phi - \frac{\Delta\lambda^2}{2}\tan\phi\right) \cong \tan^{-1}(\tan\phi) - \frac{1}{\tan^2\phi+1}\left(\frac{\Delta\lambda^2}{2}\tan\phi\right)$$

$$\tan^{-1}\left(\tan\phi - \frac{\Delta\lambda^2}{2}\tan\phi\right) \cong \phi - \frac{1}{2}\frac{\tan\phi}{\tan^2\phi+1}\Delta\lambda^2 \tag{8}$$

Substitute equation 8 into equation 7 to obtain:

$$\Delta\phi \cong \frac{1}{2}\frac{\tan\phi}{\tan^2\phi + 1}\Delta\lambda^2 \tag{9}$$

This approximate solution for the angular offset is very accurate when $\Delta\lambda$ is 1° or smaller.